① $\nabla\phi$

② $\nabla\cdot A$

③ $\nabla\times A$

④ $\nabla(fg) = f\nabla g + g\nabla f$

⑤ $\nabla(A\cdot B) = B\times(\nabla\times A) + A\times(\nabla\times B) + (B\cdot\nabla)A + (A\cdot\nabla)B$

⑥ $\nabla\cdot(fA) = f\nabla\cdot A + A\cdot(\nabla f)$

⑦ $\nabla\cdot(A\times B) = B\cdot(\nabla\times A) - A\cdot(\nabla\times B)$

⑧ $\nabla\times(fA) = f\nabla\times A + (\nabla f)\times A$

⑨ $\nabla\times(A\times B) = A(\nabla\cdot B) - B(\nabla\cdot A) + (B\cdot\nabla)A - (A\cdot\nabla)B$

⑩ $\nabla^2 f$

⑪ $\nabla\times\nabla f = 0$

⑫ $\nabla(\nabla\cdot A)$

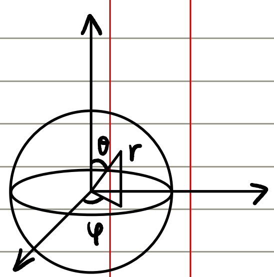
⑬ $\nabla\cdot(\nabla\times A) = 0$

⑭ $\nabla\times(\nabla\times A) = \nabla(\nabla\cdot A) - \nabla^2 A$

in spherical coordinates : (r, θ, φ)

$$g_{\mu\nu} = \begin{pmatrix} 1 & & \\ & r^2 & \\ & & r^2\sin^2\theta \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} 1 & & \\ & \frac{1}{r^2} & \\ & & \frac{1}{r^2\sin^2\theta} \end{pmatrix}$$



$$\nabla^2 f = \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{\partial^2}{\partial\varphi^2}\right)\right] f$$

最基本的：

① $\nabla_a \phi$

② $\nabla_a A^a = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} A^\mu)$

③ $\nabla \times A = \varepsilon^{abc} \partial_b A_c$

Product Rules：

$\partial_a (fg) = f \partial_a (g) + g \partial_a (f)$

④ $\nabla (fg) = f \nabla g + g \nabla f$

⑤ $\nabla (A \cdot B) = B \times (\nabla \times A) + A \times (\nabla \times B) + (B \cdot \nabla) A + (A \cdot \nabla) B$

Proof. $LHS = \partial_a (A^\mu B_\mu) = A^\mu \partial_a B_\mu + B_\mu \partial_a A^\mu$

注意： $B_\mu \partial_a A^\mu$

$= B_\mu \partial_a (g^{\mu\nu} A_\nu)$

$= B^\nu \partial_a A_\nu + B_\mu A_\nu \partial_a g^{\mu\nu}$

↑ 刚才我说这是个问题

$RHS = \varepsilon_{abc} B^b \varepsilon^{cde} \nabla_d A_e + B^b \nabla_b A_a$

$+ \varepsilon_{abc} A^b \varepsilon^{cde} \nabla_d B_e + A^b \nabla_b A_a$

注意 $\varepsilon_{abc} \varepsilon^{cde} = (-1)^2 \varepsilon_{abc} \varepsilon^{dec} = 2! \, \delta^d{}_{[a} \delta^e{}_{b]}$

所以 $RHS = 2 B^b \nabla_{[a} A_{b]} + B^b \nabla_b A_a$

$+ 2 A^b \nabla_{[a} B_{b]} + A^b \nabla_b A_a$

$= B^b \nabla_a A_b + A^b \nabla_a B_b$

注意： $B^b \nabla_a A_b + A^b \nabla_a B_b$

$= B^b \partial_a A_b - \Gamma^c_{ba} B^b A_c$

$+ A^b \partial_a B_b - \Gamma^c_{ba} A^b B_c$

而 $- \Gamma^c_{ba} B^b A_c - \Gamma^c_{ba} A^b B_c$

$= A_b B_c (- g^{cd} \Gamma^b{}_{da} - g^{bd} \Gamma^c{}_{da})$

$= -2 A_b B_c \, g^{d(c} \Gamma^{b)}{}_{da}$

$\nabla_a g^{bc} = 0 = \partial_a g^{bc} + \Gamma^b_{da} g^{dc} + \Gamma^c_{da} g^{db}$

$= 2 A_b B_c \partial_a g^{bc}$

$\therefore LHS = RHS$

⑥ $\nabla_a(fA^a) = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}\, f A^\mu)$

$= \frac{1}{\sqrt{g}} f\, \partial_\mu(\sqrt{g}\, A^\mu) + A^\mu \partial_\mu f$

$= f\, \nabla_a A^a + A^a \nabla_a f$

⑦ $\nabla\cdot(A\times B) = B\cdot(\nabla\times A) - A\cdot(\nabla\times B)$

Proof. $LHS = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}\, \varepsilon^{\mu\rho\sigma} A_\rho B_\sigma)$

$= \frac{1}{\sqrt{g}}\partial_\mu(e^{\mu\rho\sigma} A_\rho B_\sigma)$

$= \frac{1}{\sqrt{g}} e^{\mu\rho\sigma}\partial_\mu(A_\rho B_\sigma)$

$\varepsilon^{\mu\rho\sigma}$

$\mu\nu\rho\sigma\lambda\tau$

反射对称

$\varepsilon^{\mu\alpha\beta}\partial_\alpha A_\beta = \varepsilon^{\mu\alpha\beta} D_\alpha A_\beta$

$D_\alpha A_\beta = \partial_\alpha A_\beta - \Gamma^\mu_{\alpha\beta} A_\mu$

$\frac{1}{\sqrt{g}}\, \delta^{[\mu}_1 \delta^\rho_2 \delta^{\sigma]}_3$

$e^{\mu\rho\sigma}$

$sgn(\mu\rho\sigma)$

$\nabla_a (dx^\mu)_b = \partial_a (dx^\mu)_b + \Gamma^d_{ab}(dx^\mu)_d$

$\partial_a \quad (dx^\nu)_a\, \partial_\nu(\;)$

合里.

$(dx^\mu)_b = (dx^\rho)_b\, \delta^\mu{}_\rho$

$(dx^\nu)_a (dx^\rho)_b\, \partial_\nu(\delta^\mu{}_\rho) = 0$

$RHS = B_\mu \varepsilon^{\mu\alpha\beta}\partial_\alpha A_\beta - A_\mu \varepsilon^{\mu\alpha\beta}\partial_\alpha B_\beta$

$= B_\mu \varepsilon^{\mu\alpha\beta}\partial_\alpha A_\beta + A_\beta \varepsilon^{\mu\alpha\beta}\partial_\alpha B_\mu$

$= \varepsilon^{\mu\alpha\beta}(B_\mu \partial_\alpha A_\beta + A_\beta \partial_\alpha B_\mu)$

$= \varepsilon^{\mu\alpha\beta}\partial_\alpha A_\beta B_\mu$

⑧ $\nabla\times(fA) = f\,\nabla\times A + (\nabla f)\times A$

$LHS = \varepsilon^{abc}\nabla_b(f A_c)$

$= \varepsilon^{abc}\partial_b(f A_c)$

$= \varepsilon^{abc}[(\partial_b f) A_c + f\,\partial_b A_c]$

$= RHS$

⑨ $\nabla\times(A\times B) = A(\nabla\cdot B) - B(\nabla\cdot A) + (B\cdot\nabla)A - (A\cdot\nabla)B$

$LHS = \varepsilon^{abc}\nabla_b(\varepsilon_{cde} A^d B^e)$

$= \varepsilon^{abc}\partial_b(\varepsilon_{cde} A^d B^e)$

$= \left(\frac{\partial}{\partial x^\mu}\right)^a \frac{1}{\sqrt{g}} e^{\mu\nu\rho}\partial_\nu(\sqrt{g}\, e_{\rho\sigma\tau} A^\sigma B^\tau)$

$= \left(\frac{\partial}{\partial x^\mu}\right)^a \frac{1}{\sqrt{g}} (-1)^2\, 2!\, \delta^\mu_{[\sigma}\delta^\nu_{\tau]}\, \partial_\nu(\sqrt{g}\, A^\sigma B^\tau)$

$RHS = A^a\nabla_b B^b - B^a \nabla_b A^b + B^b\nabla_b A^a - A^b \nabla_b B^a$

$= \frac{1}{\sqrt{g}}\left(A^a\partial_b(\sqrt{g}\, B^b) - B^a\partial_b(\sqrt{g}\, A^b)\right)$

$+ B^b\partial_b A^a + \Gamma^a_{bc} B^b A^c - A^b\partial_b B^a - \Gamma^a_{bc} B^b A^c$

$\frac{1}{\sqrt{g}}(\sqrt{g}\, B^b)\partial_b A^a - \frac{1}{\sqrt{g}}(\sqrt{g}\, A^b)\partial_b B^a$

LHS = RHS

$\nabla_a A^a = \partial_\mu A^\mu + \Gamma^\mu_{\nu\mu} A^\nu$

$\Gamma^\mu_{\nu\mu} = \frac{1}{2} g^{\mu\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})$

$= \frac{1}{2} g^{\mu\sigma} \partial_\nu g_{\sigma\mu}$

$g^{\mu\sigma} = (g^{-1})_{\mu\sigma} = \frac{G_{\mu\rho}}{g}$ $\quad g = \det\{g_{\mu\nu}\}$

$g_{\mu\sigma} g^{\sigma\rho} = \delta_\mu{}^\rho$

$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$

$\det\{g_{\mu\nu}\} = \sum_{\mu\nu} G_{\mu\nu} g_{\mu\nu}$

$g = g + \frac{\partial g}{\partial g_{\mu\nu}} \delta g^{\mu\nu}$

$\frac{\partial g}{\partial g_{\mu\nu}} = G_{\mu\nu}$

$\frac{\partial g}{\partial x^\nu} = \frac{\partial g}{\partial g^{\mu\nu}} \frac{\partial g^{\mu\nu}}{\partial x^\nu}$

$g^{\mu\sigma} = \frac{G_{\mu\sigma}}{g}$

$\frac{1}{2} g^{\mu\sigma} \partial_\nu g_{\sigma\mu} = \frac{1}{2} \frac{G_{\mu\sigma}}{g} \partial_\nu g_{\sigma\mu}$

$= \frac{1}{2g} \partial_\nu g$ $\quad$ ~~$\partial_\nu (\ln \sqrt{g})$~~

$\nabla_a A^a = \partial_\mu A^\mu + \frac{1}{2g} (\partial_\nu g) A^\nu$

$= \frac{1}{\sqrt{g}} (\sqrt{g}\, \partial_\mu A^\mu + \partial_\nu (\sqrt{g}) A^\nu$

$= \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} A^\mu)$

Second Derivatives

⑩ $\nabla^2 f = \nabla^a \nabla_a f = \frac{1}{\sqrt{g}}\partial_a(\sqrt{g}\,\partial^a f)$

⑪ $\nabla \times \nabla f = \varepsilon^{abc}\partial_b \partial_c f = 0$

⑫ $\nabla(\nabla\cdot A) = \nabla_a \nabla_b A^b = \partial_a\left(\frac{1}{\sqrt{g}}\partial_b(\sqrt{g}A^b)\right)$

⑬ $\nabla\cdot(\nabla\times A) = \nabla_a(\varepsilon^{abc}\nabla_b A_c)$

$= \nabla_a(\varepsilon^{abc}\partial_b A_c)$

$= \frac{1}{\sqrt{g}}\partial_a(\sqrt{g}\,\varepsilon^{abc}\partial_b A_c)$

$= \varepsilon^{abc}\partial_a\partial_b A_c = 0$

⑭ $\nabla\times(\nabla\times A) = \nabla(\nabla\cdot A) - \nabla^2 A$

Proof. LHS $= \varepsilon^{abc}\nabla_b(\varepsilon_{cde}\nabla^d A^e)$ $\quad g_{cf}\,\varepsilon^{fde}\partial_d A_e$

$= \varepsilon^{abc}\partial_b(\varepsilon_{cde}\partial^d A^e)$

$= \frac{1}{\sqrt{g}}(-1)^2\, 2!\;\delta^a_{[d}\delta^b_{e]}\;\partial_b(\sqrt{g}\,\partial^d A^e)$

$= \frac{1}{\sqrt{g}}\,2!\;\partial_b(\sqrt{g}\,\partial^{[a}A^{b]})$

$\varepsilon_{cde}\nabla^d A^e = \frac{1}{2}\varepsilon_{cde}(\nabla^d A^e - \nabla^e A^d)$

$\partial^d A^e + g^{df}T^e_{fg}A^g$

$- \partial^e A^d - g^{ef}T^d_{fg}A^g$

$\neq \varepsilon_{cde}\partial^d A^e$

RHS $= \nabla^a\nabla_b A^b - \nabla^b\nabla_b A^a$

$= \partial^a\left(\frac{1}{\sqrt{g}}\partial_b(\sqrt{g}A^b)\right)$

$- \partial_b(\nabla^b A^a) - T^b_{bc}\nabla^c A^a - T^a_{bc}\nabla^b A^c$

$= \partial^a\left(\frac{1}{\sqrt{g}}\partial_b(\sqrt{g}A^b)\right) - \frac{1}{\sqrt{g}}\partial_b(\sqrt{g}\,\nabla^b A^a)$

$- T^a_{bc}\nabla^b A^c$

$\left(\partial^a\partial_b A^b + \partial^a\left(A^b\frac{\partial_b\sqrt{g}}{\sqrt{g}}\right)\right)$

$\frac{1}{\sqrt{g}}\partial_b(\sqrt{g}\,\partial^a A^b) - (\partial^a A^b)\frac{\partial_b\sqrt{g}}{\sqrt{g}}$

$\frac{1}{\sqrt{g}}\partial_b(\sqrt{g}\,\partial^a A^b) + A^b\partial^a\left(\frac{\partial_b\sqrt{g}}{\sqrt{g}}\right)$

$-\frac{1}{\sqrt{g}}\partial_b(\sqrt{g}\,\partial^b A^a) - \frac{1}{\sqrt{g}}\partial_b(\sqrt{g}\,g^{bc}T^a_{cd}A^d)$

$= \frac{1}{\sqrt{g}}\,2\,\partial_b(\sqrt{g}\,\partial^{[a}A^{b]}) + A^b\partial^a\left(\frac{\partial_b\sqrt{g}}{\sqrt{g}}\right)$

$-\frac{1}{\sqrt{g}}\partial_b(\sqrt{g}\,g^{bc}T^a_{cd}A^d) - T^a_{bc}\nabla^b A^c$

显然，下面我们需要证明：

$A^b\partial^a\left(\frac{\partial_b\sqrt{g}}{\sqrt{g}}\right) - \frac{1}{\sqrt{g}}\partial_b(\sqrt{g}\,g^{bc}T^a_{cd}A^d) - T^a_{bc}\nabla^b A^c = 0$ ✗

$$\frac{1}{\sqrt{g}}\partial_b(\sqrt{g}\, g^{bc}\Gamma^a_{cd}A^d) = A^b\partial^a\left(\frac{\partial_b\sqrt{g}}{\sqrt{g}}\right) - \Gamma^a_{bc}\nabla^b A^c$$

Proof. $LHS = g^{bc}\Gamma^a_{cd}\partial_b A^d + A^d\frac{1}{\sqrt{g}}\partial_b(\sqrt{g}\, g^{bc}\Gamma^a_{cd})$ ✗

$= \Gamma^a_{bc}\partial^b A^c + A^d\frac{1}{\sqrt{g}}\partial_b(\sqrt{g}\, g^{bc}\Gamma^a_{cd})$

$RHS = A^b\partial^a\left(\frac{\partial_b\sqrt{g}}{\sqrt{g}}\right) - \Gamma^a_{bc}\partial^b A^c - \Gamma^a_{bc}g^{bd}\Gamma^c_{de}A^e$

$LHS = \varepsilon^{abc}\partial_b(g_{cf}\varepsilon^{fde}\partial_d A_e)$ $\quad g^{ge}\partial^d A_g$

$= \varepsilon^{abc}\partial_b(\varepsilon_{cde}\, g^{fd}g^{ge}\partial_f A_g)$

$= \varepsilon^{abc}\partial_b(\varepsilon_{cde}(\partial^d A^e - A_g\partial^d g^{ge}))$

$= \varepsilon^{abc}\partial_b(\varepsilon_{cde}\partial^d A^e) - \varepsilon^{abc}\partial_b(\varepsilon_{cde}A_g\partial^d g^{ge})$

$= \frac{1}{\sqrt{g}}\, 2!\,\partial_b(\sqrt{g}\,\partial^{[a}A^{b]}) - \varepsilon^{abc}\partial_b(\varepsilon_{cde}A_g\partial^d g^{ge})$

$-\frac{1}{\sqrt{g}}(-1)^2\, 2!\,\delta^a_{[d}\delta^b_{e]}\partial_b(\sqrt{g}\, A_g\partial^d g^{ge}) = -\frac{1}{\sqrt{g}}\, 2\,\partial_b(\sqrt{g}\, A_c\partial^{[a}g^{b]c})$

$-g^{ce}g^{ab}\partial^d g_{bc} = \partial^d g^{ae}$

$\partial^d(g^{ab}g_{bc}) = 0 = g^{ab}\partial^d g_{bc} + g_{bc}\partial^d g^{ab}$

$\Rightarrow \partial^d g^{ab} = -g^{bc}g^{ae}\partial^d g_{ce}$

下面要证明.

$$-\frac{1}{\sqrt{g}}\, 2\,\partial_b(\sqrt{g}\, A_c\partial^{[a}g^{b]c}) = A^b\partial^a\left(\frac{\partial_b\sqrt{g}}{\sqrt{g}}\right) - \frac{1}{\sqrt{g}}\partial_b(\sqrt{g}\, g^{bc}\Gamma^a_{cd}A^d) - \Gamma^a_{bc}\nabla^b A^c$$

$\frac{\partial_b\sqrt{g}}{\sqrt{g}} = \frac{1}{2}g^{cd}\partial_b g_{cd}$

$LHS = -\frac{1}{\sqrt{g}}\, 2\,\partial_b(\sqrt{g}\, A_c\partial^{[a}g^{b]c})$

$\partial^a g^{bc} - \partial^b g^{ac} = -g^{ad}g^{be}g^{cf}(\partial_d g_{ef} - \partial_e g_{df}) = g^{be}g^{cf}(2\Gamma^a_{fe} - g^{ad}\partial_f g_{ed})$

$\Gamma^a_{bc} = g^{ad}\frac{1}{2}(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc})$

$\Gamma^a_{fe} = g^{ad}\frac{1}{2}(\partial_f g_{ed} + \partial_e g_{fd} - \partial_d g_{fe})$ $\quad 2\Gamma^a_{fe} - g^{ad}\partial_f g_{ed} = g^{ad}(\partial_e g_{fd} - \partial_d g_{fe})$

$LHS = -\frac{1}{\sqrt{g}}\partial_b(\sqrt{g}\, A_f g^{be}(2\Gamma^a_{fe} - g^{ad}\partial_f g_{ed}))$

$= -\frac{1}{\sqrt{g}}\, 2\,\partial_b(\sqrt{g}\, A_f g^{be}\Gamma^a_{fe}) + \frac{1}{\sqrt{g}}\partial_b(\sqrt{g}\, A_f g^{be}g^{ad}\partial_f g_{ed})$

LHS − RHS

$$= -\frac{1}{\sqrt{g}}\partial_b(\sqrt{g}\, g^{bc} T^a_{cd} A^d) + \frac{1}{\sqrt{g}}\partial_b(\sqrt{g}\, A^f g^{be} g^{ad} \partial_f g_{ed}) - A^b \partial^a(\tfrac{1}{2} g^{cd}\partial_b g_{cd}) + T^a_{bc}\nabla^b A^c$$

$$= -A^d \frac{1}{\sqrt{g}}\partial_b(\sqrt{g}\, g^{bc} T^a_{cd}) + \frac{1}{\sqrt{g}} A^e \partial_b(\sqrt{g}\, g^{bd} g^{ac}\partial_e g_{dc}) - A^b\partial^a(\tfrac{1}{2}g^{cd}\partial_b g_{cd}) + A^e g^{bd} T^a_{bc} T^c_{de}$$

$$- g^{bc}T^a_{cd}\partial_b A^d + g^{bd}g^{ac}(\partial_e g_{dc})(\partial_b A^e) + T^a_{bc}\partial^b A^c$$

现在需要证明的是：

① $-\frac{1}{\sqrt{g}}\partial_b(\sqrt{g}\, g^{bc}T^a_{ce}) + \frac{1}{\sqrt{g}}\partial_b(\sqrt{g}\, g^{bd}g^{ac}\partial_e g_{dc}) - \partial^a(\tfrac{1}{2}g^{cd}\partial_e g_{cd}) + g^{bd}T^a_{bc}T^c_{de} = 0$

② $-g^{bc}T^a_{ce} + g^{bd}g^{ac}(\partial_e g_{dc}) + T^a_{de}g^{db} = 0$

又错了

⑭ 此推导其实很简单，只需要注意 $\nabla_a \varepsilon^{bcd} = 0$

Pf. $\nabla_a \varepsilon^{b_1\cdots b_n}$

$$= (dx^\mu)_a \left(\frac{\partial}{\partial x^{\nu_1}}\right)^{b_1}\cdots\left(\frac{\partial}{\partial x^{\nu_n}}\right)^{b_n}\left[\partial_\mu\left(\frac{1}{\sqrt{g}}\right)\mathrm{sgn}\begin{pmatrix}\nu_1\cdots\nu_n\\ 1\cdots n\end{pmatrix} + \frac{1}{\sqrt{g}}\left(\Gamma^{\nu_1}_{\mu\rho}\,\mathrm{sgn}\begin{pmatrix}\rho\,\nu_2\cdots\nu_n\\ 1\;2\cdots n\end{pmatrix} + \cdots + \Gamma^{\nu_n}_{\mu\rho}\,\mathrm{sgn}\begin{pmatrix}\nu_1\cdots\nu_{n-1}\,\rho\\ 1\cdots n-1\;n\end{pmatrix}\right)\right]$$

$$= (dx^\mu)_a\left(\frac{\partial}{\partial x^{\nu_1}}\right)^{b_1}\cdots\left(\frac{\partial}{\partial x^{\nu_n}}\right)^{b_n}\left[-\frac{1}{2}g^{-\frac{3}{2}}(\partial_\mu g)\,\mathrm{sgn}\begin{pmatrix}\nu_1\cdots\nu_n\\ 1\cdots n\end{pmatrix} + \frac{1}{2}g^{-3/2}(\partial_\mu g)\,\mathrm{sgn}\begin{pmatrix}\nu_1\cdots\nu_n\\ 1\cdots n\end{pmatrix}\right]$$

$$= 0$$

↓ $\Gamma^\rho_{\mu\rho}\,\mathrm{sgn}\begin{pmatrix}\nu_1\cdots\nu_n\\ 1\cdots n\end{pmatrix}$

↓ $\frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\rho\sigma} + \partial_\rho g_{\mu\sigma} - \partial_\sigma g_{\mu\rho})$

$= \frac{1}{2}g^{\rho\sigma}\partial_\mu g_{\rho\sigma}$

$= \frac{1}{2}\frac{1}{g}\partial_\mu g$

$$\nabla\times(\nabla\times A) = \nabla(\nabla\cdot A) - \nabla^2 A$$

$$\text{LHS} = \varepsilon_{abc}\nabla^b(\varepsilon^{cde}\nabla_d A_e)$$

$$= \varepsilon_{abc}\varepsilon^{cde}\nabla^b\nabla_d A_e$$

$$= (-1)^2\varepsilon_{abc}\varepsilon^{dec}\nabla^b\nabla_d A_e$$

$$= (-1)^2\, 2!\,\delta^d{}_{[a}\delta^e{}_{b]}\nabla^b\nabla_d A_e$$

$$= (-1)^2\, 2!\,\nabla^b\nabla_{[a}A_{b]}$$

$$= \nabla^b\nabla_a A_b - \nabla^b\nabla_b A_a = \nabla_a\nabla^b A_b - \nabla^b\nabla_b A_a + R_a{}^b A_b$$

$$2g^{bc}\nabla_{[b}\nabla_{a]}A_c = g^{bc}R_{bac}{}^d A_d = R_a{}^d A_d$$

$$\nabla^b\nabla_a A_b - \nabla_a\nabla^b A_b = R_a{}^b A_b$$